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Note on separate continuity and the Namioka property

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Abstract

A pair $\langle B, K\rangle$ is a *Namioka pair* if K is compact and for any separately continuous $f : B \times K \to \mathbb{R}$, there is a dense $A \subseteq B$ such that f is (jointly) continuous on $A \times K$. We give an example of a Choquet space B and separately continuous $f : B \times \beta B \to \mathbb{R}$ such that the restriction $f|_\Delta$ to the diagonal does not have a dense set of continuity points. However, for K a compact fragmentable space we have: For any separately continuous $f : T \times K \to \mathbb{R}$ and for any Baire subspace F of $T \times K$, the set of points of continuity of $f|_F : F \to \mathbb{R}$ is dense in F. We say that $\langle B, K\rangle$ is a *weak-Namioka pair* if K is compact and for any separately continuous $f : B \times K \to \mathbb{R}$ and a closed subset F projecting irreducibly onto B, the set of points of continuity of $f|_F$ is dense in F. We show that T is a Baire space if the pair $\langle T, K\rangle$ is a weak-Namioka pair for every compact K. Under (CH) there is an example of a space B such that $\langle B, K\rangle$ is a Namioka pair for every compact K but there is a countably compact C and a separately continuous $f : B \times C \to \mathbb{R}$ which has no dense set of continuity points; in fact, f does not even have the Baire property.

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1. Introduction

The celebrated Namioka theorem [12] asserts that for any separately continuous function $f : B \times K \to \mathbb{R}$ on the product of a Čech complete space B and a compact space K, there is a dense set $A \subseteq B$ such that f is jointly continuous at each point of the set $A \times K$. (Any such set A actually extends to a G_δ-set A' with the same property.)

We shall call any pair $\langle B, K\rangle$ with B completely regular and K compact, satisfying the assertion of this Namioka theorem, a *Namioka pair*. (In the terminology of [11, Definition 2.1], the pair is in the relation $\mathcal{N}(B, K)$.)

In the sequel we shall consider only completely regular spaces. Saint Raymond [14] proved that if B is such that $\langle B, K\rangle$ is a Namioka pair for any compact K, then B is a Baire space. Haydon [6] showed that there are Baire spaces, even Choquet spaces [7, 8.12], B and compact scattered spaces K such that $\langle B, K\rangle$ are not Namioka pairs. Recall that a space X is a *Choquet space* if Player II has a winning strategy for the *Choquet game* [7, 8.10] on X: Players I and II alternately choose nonempty open sets $U_0, U_1, \ldots$ (by Player I) and $V_0, V_1, \ldots$ (by Player II) such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots$. Player II wins this game if $\bigcap_n V_n = \bigcap U_n \neq \emptyset$. Every Choquet space is a Baire space.

Using a construction from [2] one can exhibit Choquet spaces B such that the pair $\langle B, \beta B\rangle$ fails the Namioka property rather dramatically:

Theorem 1.1. *There is a Choquet space B and a separately continuous function $f : B \times \beta B \to \mathbb{R}$ such that the restriction $f|_\Delta$ to the diagonal $\Delta = \{(b, b): b \in B\}$ does not have the Baire property.*

Let us recall that a function on a Baire space has the Baire property [10, Section 32, I] exactly when it is continuous apart from a meager set. In particular, a function with a dense set of continuity points has the Baire property. On the other hand, for the important class of compact fragmentable spaces [13], which includes all compact scattered spaces, one can obtain the following result (closely related to some results by Kenderov, Kortezov and Moors [8,9]):

Theorem 1.2. *For any separately continuous function $f : T \times K \to \mathbb{R}$, on the product of a completely regular space T and a compact fragmentable space K, and any Baire subspace B of $T \times K$, the set of points of continuity of the restriction $f|_B : B \to \mathbb{R}$ is dense in B.*

Using an argument from [12], one can see that for a Baire space B and a compact space K the Namioka property of the pair $\langle B, K\rangle$ is equivalent to the following property: For any separately continuous $f : B \times K \to \mathbb{R}$ and a closed subset F of $B \times K$ projecting irreducibly onto B, the set of points in F at which f is jointly continuous is dense in F.

Let us say that $\langle B, K\rangle$ is a *weak-Namioka pair*, if K is compact and for any separately continuous $f : B \times K \to \mathbb{R}$ and a closed subset F of $B \times K$ projecting irreducibly onto B, the set of points of continuity of the restriction $f|_F : F \to \mathbb{R}$ is dense in F.

Now, Theorem 1.1 shows that some pairs $\langle B, \beta B\rangle$, with B Baire, may not be weak-Namioka pairs. One can also show that for any infinite compact F-space K there is a

Choquet space B of the same weight such that $\langle B, K\rangle$ is not a weak-Namioka pair. In case of $K = \beta\mathbb{N}$, this strengthens a result from [8].

By Theorem 1.2, each $\langle B, K\rangle$ with B Baire and K compact fragmentable is a weak-Namioka pair. In particular, the Haydon examples provide pairs $\langle B, K\rangle$, with B Baire and K compact scattered, which are weak Namioka pairs but fail to be Namioka pairs.

The Saint Raymond theorem can be strengthened (with essentially the same reasoning) to the following effect:

Proposition 1.3. *If T is a completely regular space such that for any compact K, $\langle T, K\rangle$ is a weak-Namioka pair, then T is a Baire space.*

It would be interesting to clarify if, for a space B, the property that each pair $\langle B, K\rangle$ with K compact is weakly Namioka, implies that B is a Namioka space, i.e., if every $\langle B, K\rangle$ is in fact a Namioka pair.

For many standard examples of Namioka spaces B, these spaces have, in fact, the following stronger property: the assertion of the Namioka theorem is true for any separately continuous function $f : B \times C \to \mathbb{R}$ with C being an arbitrary countably compact space.

However, at least consistently, this property is essentially stronger than the Namioka property of a Baire space:

Example 1.4. Let $B = (D^{\aleph_1})_\delta$ be the product of $\aleph_1$ copies of $\{0, 1\}$ equipped with the G_δ topology. Then, there is a countably compact space C and a separately continuous $f : B \times C \to \mathbb{R}$ which does not have the Baire property. However, under (CH), B is a Namioka space.

The details of this example will be verified in Section 5. We shall also include in this section a remark related to some results obtained by Maxim Burke [3], concerning Borel measurability of separately continuous functions.

2. Proof of Theorem 1.1

A compact space K is an *F-space* if any continuous map $c : U \to [0, 1]$, defined on an open σ-compact set $U \subseteq K$, can be extended continuously over K.

The following fact can be derived from the results in [2].

Lemma 2.1. *For each infinite compact F-space K there is a Choquet space B, a separately continuous function $u : B \times K \to \mathbb{R}$ and a continuous function $v : B \to K$ such that the function $w : B \to \mathbb{R}$, $w(b) = u(b, v(b))$, does not have the Baire property.*

We shall outline the construction, concentrating on the Čech–Stone compactification $\beta\mathbb{N}$ of natural numbers. Let $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

Proof. Let M be the space of all sequences $(c_1, c_2, \ldots)$ with $c_i : A \to \{0, 1\}$, $A \subseteq \mathbb{N}$ and c_{i+1} extending c_i. M is equipped with the subspace topology inherited from a product of discrete spaces.

Let C be the space of all functions $s:\mathbb{N}\to\{0,1\}$ with the topology of pointwise convergence on $\beta\mathbb{N}$; i.e., $s_\sigma\to s_0$ in C, if for any $x\in\beta\mathbb{N}$, $s^\beta_\sigma(x)\to s^\beta_0(x)$, where the superscript β denotes the continuous extension over $\beta\mathbb{N}$.

Then, the space B is the subspace of the product $\mathbb{N}^*\times C\times M$ consisting of all sequences $b=(x,s,c_1,c_2,\ldots)$ with s extending each c_i, and x not in the union of the closures in $\beta\mathbb{N}$ of the domains of c_i.

The reasoning in Section 2 of [2] shows that B is a Choquet space and the set

$$E=\bigl\{(x,s,c_1,c_2,\ldots)\in B\colon\ s^\beta(x)=1\bigr\} \tag{1}$$

fails the Baire property (i.e., E is not open modulo meager sets in B).

Let us define $u:B\times\beta\mathbb{N}\to\{0,1\}$ and $v:B\to\beta\mathbb{N}$ by the formulas

$$u\bigl((x,s,c_1,c_2,\ldots),y\bigr)=s^\beta(y),\qquad v(x,s,c_1,c_2,\ldots)=x.$$

Then, for $w(b)=u(b,v(b))$ we get

$$w(x,s,c_1,c_2,\ldots)=s^\beta(x).$$

The topology in C guarantees that u is separately continuous, and v is the projection onto the first coordinate. However w does not have the Baire property, as the set E in (1) fails this property and $E=w^{-1}(1)$. □

Remark 2.2. In the outlined construction, the weight of B is equal to the cardinality of $\beta\mathbb{N}$, and hence greater than the weight of $\beta\mathbb{N}$. To lower the weight of B one should consider a topology in the set C stronger than the pointwise topology, described in [2, Section 2]. In fact, the construction in [2] allows one to choose in Lemma 2.1 a space B with weight equal to the weight of K.

Proof of Theorem 1.1. Let $u:B\times K\to\mathbb{R}$, $v:B\to K$ be the functions described in the assertion of Lemma 2.1 (one may consider $K=\beta\mathbb{N}$ and the space B described above.)

Let $v^\beta:\beta B\to K$ be the continuous extension of v over the Čech–Stone compactification, and let $f:B\times\beta B\to\mathbb{R}$ be defined by

$$f(b,z)=u\bigl(b,v^\beta(z)\bigr).$$

Then f is separately continuous and its restriction to the diagonal $\Delta=\{(b,b)\colon\ b\in B\}$ is given by $f(b,b)=u(b,v(b))$; i.e., the restriction $f|_\Delta$ coincides with the function w in Lemma 2.1. Therefore, $f|_\Delta$ fails to have the Baire property. □

3. Proof of Theorem 1.2

By a result of [15, Corollary 1.1], there is a metric d on K, fragmenting K, such that the identity $K_d\to K$ is continuous, where K_d is endowed with the topology generated by d. We shall write

$$K(z,\varepsilon)=\bigl\{y\in K\colon\ d(z,y)<\varepsilon\bigr\},\quad z\in K. \tag{2}$$

Let τ be the topology of the product $T\times K$, and τ_d the topology of the product $T\times K_d$. We shall check the following:

Claim A. *There is a comeager set G in B such that the restriction $f|_G$ is continuous and for each point in G the neighborhoods with respect to $(B, \tau|_B)$ and $(B, \tau_d|_B)$ are the same.*

To begin with, we shall check, following a reasoning in the proof of [15, Proposition 2.5] that

$$\text{the set of points in } B \text{ where the neighborhoods in } (B, \tau|_B) \text{ and } (B, \tau_d|_B) \text{ coincide is comeager in } (B, \tau|_B). \tag{3}$$

Indeed, let C_n be the union of all $H \in \tau|_B$ which are contained in some rectangle $E \times F$ with E open in T and d-diam $F \leqslant 1/n$. The set of points described in (3) is the intersection $\bigcap_{n=1}^{\infty} C_n$, and hence, B being Baire, it is enough to make sure that each C_n is dense in B. Aiming at a contradiction, assume that $D = B \setminus \overline{C_n} \neq \emptyset$, and let D' be the projection of D onto K. Since d fragments K, there is W, open in K, with $W \cap D' \neq \emptyset$ and d-diam$(W \cap D') \leqslant 1/n$. Then, with $F = W \cap D'$, some point $(x, y) \in D$ is in the interior of $B \cap (T \times F)$ in $(B, \tau|_B)$. We obtained $(x, y) \in C_n \cap D$, which is impossible.

Having checked (3), let us notice that, the identity $T \times K_d \to T \times K$ being continuous, the function f is also separately continuous with respect to the topology τ_d. The factor K_d is metrizable, and hence, by W. Rudin's extension of the classical Baire's theorem [16], we conclude that f is of the first Baire class with respect to τ, and in particular, for each Baire subspace Z of $T \times K_d$ there is a comeager set G in Z such that $f|_G$ is continuous. Taking as Z the set described in (3), we obtain G satisfying the assertion of Claim A.

Since B is Baire, the assertion of the theorem is equivalent to the following.

Claim B. *For each $\delta > 0$, the set $\bigcup\{W \in \tau|_B\colon \operatorname{diam} f(W) \leqslant \delta\}$ is dense in B.*

In fact, replacing B by any of its open nonempty subsets, it is enough to find a nonempty $W \in \tau|_B$ with diam $f(W) \leqslant \delta$. To that end, for each (x, y) in the set G described in Claim A, we shall fix a natural number $n(x, y)$ such that

$$\bigl|f(x, y) - f(x, y')\bigr| \leqslant \frac{\delta}{6} \quad \text{if } d(y, y') < \frac{1}{n(x, y)}. \tag{4}$$

This is possible, as f is continuous on the variable y, and the identity $K_d \to K$ is continuous. Let

$$G_n = \bigl\{(x, y) \in G\colon n(x, y) = n\bigr\}. \tag{5}$$

Now, $G = \bigcup_n G_n$ and since G is comeager in B, there is an n and there is a nonempty open set W in B such that

$$W \subseteq \overline{W \cap G_n}. \tag{6}$$

Let us fix $(x_0, y_0) \in W \cap G$. Since $f|_G$ is continuous at (x_0, y_0), shrinking W if necessary, we can assume that

$$\bigl|f(x, y) - f(x_0, y_0)\bigr| \leqslant \frac{\delta}{6} \quad \text{for } (x, y) \in W \cap G. \tag{7}$$

Finally, taking into account that neighborhoods of (x_0, y_0) with respect to $\tau|_B$ and $\tau_d|_B$ coincide, we can replace W by a smaller neighborhood to also guarantee that

$$(x_0, y_0) \in W \subseteq T \times K\left(y_0, \frac{1}{2n}\right). \tag{8}$$

Let

$$M = \mathrm{proj}_B(W \cap G_n) \tag{9}$$

be the projection parallel to K. Then, by (6) and (8),

$$W \subseteq \overline{M} \times K\left(y_0, \frac{1}{2n}\right). \tag{10}$$

Let us check that

$$\left|f(x, y) - f(x_0, y_0)\right| \leqslant \frac{\delta}{3} \quad \text{for } x \in M,\ y \in K\left(y_0, \frac{1}{2n}\right). \tag{11}$$

Indeed, for (x, y) in (11), there is, by (9), a point $(x, y') \in W \cap G_n$. Then, by (10), $d(y, y') < 1/n$, and $|f(x, y) - f(x, y')| \leqslant \delta/6$ by (4); cf. (5). This, combined with (7), gives (11).

Now, with $y \in K(y_0, 1/(2n))$, the continuity of f on variable x, extends (11) over $(x, y) \in \overline{M} \times K(y_0, 1/(2n))$, and by (10), diam $f(W) \leqslant \delta$. This ends the proof of Claim B, and completes the proof of the theorem.

Remark 3.1. Theorem 1.2 is closely related to a theorem by Kenderov, Kortezov and Moors, [9, Theorem 11], and in case $B = T \times K$ it follows from this theorem. However, we did not see how to derive the general case from this theorem, even if T is a Baire space and B is the graph of a continuous function from T to K.

A Banach space E is an Asplund space if the unit ball $B(E^*)$ in the dual space with the weak* topology is norm fragmented, [4], Theorem 5.2. There are Asplund spaces E which fail the Namioka property, i.e., there is $f : B \to E$ with B Baire, continuous with respect to the weak topology but without continuity points with respect to the norm topology; cf. Haydon [6].

We have, however, the following fact, where w and w^* stand respectively for the weak and weak* topology.

Corollary 3.2. *Let E be an Asplund space and let $B(E^*)$ be the unit ball in the dual space. Then, for any continuous map $(u, v) : B \to (E, w) \times (B(E^*), w^*)$, defined on a Baire space B, the map $b \mapsto \langle u(b), v(b)\rangle$ has a dense set of continuity points ($\langle\,,\,\rangle : E \times E^* \to \mathbb{R}$ is the duality map, $\langle x, y^*\rangle = y^*(x)$).*

Proof. Let $f : B \times (B(E^*), w^*) \to \mathbb{R}$ be defined by $f(b, x^*) = \langle u(b), x^*\rangle$, and let $F = \{(b, v(b)) \colon b \in B\} \subseteq B \times (B(E^*), w^*)$. Then f is separately continuous and F is a closed set such that the projection maps F homeomorphically onto B. Also, $\langle u(b), v(b)\rangle = f(u(b), v(b))$, hence the map in the assertion can be identified with the restriction of f to F. In effect, the assertion follows from Theorem 1.2. □

4. Proof of Proposition 1.3

Following closely a reasoning of Saint Raymond [14, Proof of Theorem 3], we shall establish the following fact, from which Proposition 1.3 follows readily.

Proposition 4.1. *Let X be a completely regular space of first category. Then there is a separately continuous function $\phi : X \times \beta X \to [0,1]$ whose restriction to the diagonal $\phi|_\Delta : \Delta \to [0,1]$ is discontinuous at each point.*

We start from a lemma parallel to Saint Raymond's Lemma 4, and the proof will be a minor modification of his reasoning.

Lemma 4.2. *Let F be a closed meager set in a Tychonoff space X. There exists a compact space K, separately continuous $f : X \times K \to I$ and continuous $g : X \to K$ such that $u(x) = f(x, g(x))$ takes on only the values 0 and 1, $u(x) = 0$ for $x \in F$ and $u(x) = 1$ for x in an open set dense in X.*

Proof. Let $(\phi_\lambda)_{\lambda\in\Lambda}$ be a maximal collection of non-zero continuous functions $\phi_\lambda : X \to [0,1]$ whose supports $\overline{\phi_\lambda(0,1]}$, $\lambda \in \Lambda$, are pairwise disjoint and miss the set F.

Let $\omega\Lambda = \lambda \cup \{\omega\}$ be the one-point compactification of λ with the discrete topology, and let

$$\pi : \omega\Lambda \times I \to K$$

be the quotient mapping matching the set $(\omega\Lambda \times \{0\}) \cup (\{\omega\} \times I)$ to a point.

We define $f : X \times K \to I$ by the formula

$$f\big(x, \pi(\gamma, t)\big) = \begin{cases} 0 & \text{if } t = 0 \text{ or } \gamma = \omega, \\ \frac{2t\cdot\phi_\gamma(x)}{t^2+\phi_\gamma^2(x)} & \text{if } t \neq 0 \text{ and } \gamma \in \Lambda, \end{cases}$$

and let $g : X \to K$ be defined by

$$g(x) = \begin{cases} \pi(\lambda, \phi_\lambda(x)) & \text{if } \phi_\lambda(x) > 0, \\ \pi(\omega, 0) & \text{if } x \notin \bigcup_{\lambda\in\Lambda} \phi_\lambda^{-1}((0,1]). \end{cases}$$

Let $u(x) = f(x, g(x))$. If $\phi_\lambda(x) > 0$, then $u(x) = f(x, \pi(\lambda, \phi_\lambda(x))) = \frac{2\phi_\lambda^2(x)}{\phi_\lambda^2(x)+\phi_\lambda^2(x)} = 1$, and if $x \notin \bigcup_{\lambda\in\Lambda} \phi_\lambda^{-1}((0,1])$, $u(x) = 0$. □

Proof of Proposition 4.1. Let $X = \bigcup_{i=1}^\infty F_i$ with $F_1 \subseteq F_2 \subseteq \cdots$ closed and meager. Let $f_i : X \times K_i \to I$, $g_i : X \to K_i$ be as in Lemma 4.2, and let $u_i(x) = f_i(x, g_i(x))$.

We set $K = K_1 \times K_2 \times \cdots$ and define $f : X \times K \to I$, $g : X \to K$ by

$$f(x, y_1, y_2, \ldots) = \sum_{i=1}^\infty 2^{-i} f_i(x, y_i), \qquad g(x) = \big(g_1(x), g_2(x), \ldots\big).$$

Then

$$u(x) = f\big(x, g(x)\big) = \sum_{i=1}^\infty 2^{-i} u_i(x).$$

Let $x \in X$ and let F_i be the first set containing x. Then, $u(x) = \sum_{j=1}^{i-1} 2^{-j}$, but in each neighborhood of x there is a point x' with $u(x') = (\sum_{j=1}^{i-1} 2^{-j}) + 2^{-i}$. Now, to complete the proof, let us consider the continuous extension $g^\beta : \beta X \to K$ and let us define $\phi : X \times \beta X \to [0, 1]$ by

$$\phi(x, z) = f\big(x, g^\beta(z)\big).$$

Then ϕ is separately continuous and $\phi(x, x) = f(x, g(x)) = u(x)$. Hence the restriction $\phi|_\Delta$ is discontinuous at each point of Δ. $\square$

Remark 4.3. Let $\langle B, K\rangle$ be a pair of spaces, with B Baire and K compact, which is not weak-Namioka. Then there is a Baire space F contained in $B \times K$ which maps onto B by a perfect irreducible map, and a separately continuous function $f : B \times K \to \mathbb{R}$ such that the set of continuity points of the restriction $f|_F$ is not dense in F. If $h : F \to K$ is the projection map and $h^\beta : \beta F \to K$ is the Čech–Stone extension of h, define $\phi : F \times \beta F \to \mathbb{R}$ by $\phi((b, y), z) = f(b, h^\beta(z))$. Observe that, for $(b, y) \in F$, $\phi|_\Delta((b, y), (b, y)) = f|_F(b, y)$, so that the continuity points of the restriction $\phi|_\Delta$ are not dense in the diagonal.

5. Verification of Example 1.4 and a remark on Borel measurability of separately continuous functions

We shall denote by $C_p(X)$ the space of continuous functions $f : X \to \mathbb{R}$ equipped with the pointwise topology.

The space B in Example 1.4 is a P-space without isolated points. Therefore the first part of Example 1.4 follows immediately from the following

Proposition 5.1. *Let B be a Baire P-space without isolated points and let $C = \{u \in C_p(B)\colon u : B \to \{0, 1\}\}$. Then C is countably compact and the evaluation map $e : B \times C \to \{0, 1\}$ does not have the Baire property.*

Proof. To see that the space C is countably compact, let us consider any countable infinite set A in C. There is a clopen partition $\mathcal{P}$ of B such that each function in A is constant on every member of $\mathcal{P}$. If $u \in \overline{A}$ (pointwise closure in the Tychonoff product $\{0, 1\}^B$) then u is constant on elements of $\mathcal{P}$, and hence $u \in C$. Hence the closure of A in C is compact—showing that C is countably compact. (It is worth a remark here that we have actually shown that C is ω-bounded; that is, the closure of every countable subset of C is compact. This is stronger than being countably compact.)

To check that the evaluation map

$$e(x, u) = u(x)$$

fails the Baire property it is enough to show that for any nonempty clopen rectangle $U \times W$ in $B \times C$, and any sequence $F_1 \subset F_2 \subset \cdots$ of closed sets in $B \times C$ with empty interiors, the evaluation e is not constant on $U \times W \setminus \bigcup_{i=1}^\infty F_i$.

To that end, we shall define inductively collections $\mathcal{G}_m$ and functions φ_m on $\mathcal{G}_m$ such that

(1) $\mathcal{G}_m$ is a disjoint family of nonempty clopen sets in B and $\varphi_m(G)$, $G \in \mathcal{G}_m$, is a nonempty clopen set in C,
(2) $\mathcal{G}_m$ refines $\mathcal{G}_{m-1}$ and $\bigcup \mathcal{G}_m$ is dense in U,
(3) $G \times \varphi_m(G) \subset U \times W \setminus F_m$, for $G \in \mathcal{G}_m$,
(4) $\varphi_{m+1}(G) \subset \varphi_m(H)$, whenever $G \subset H$, $G \in \mathcal{G}_{m+1}$, $H \in \mathcal{G}_m$.

Let us start with $\mathcal{G}_0 = \{U\}$, $\varphi_0(U) = W$, letting $F_0 = \emptyset$. Then, given $G \in \mathcal{G}_m$, let $\mathcal{H}_G$ be a maximal disjoint collection of nonempty clopen subsets of G such that for any $H \in \mathcal{H}_G$ there is a nonempty clopen $\varphi_G(H)$ in C with $H \times \varphi_G(H) \subset G \times \varphi_m(G) \setminus F_{m+1}$. We let $\mathcal{G}_{m+1}$ be the union of the collections $\mathcal{H}_G$ with $G \in \mathcal{G}_m$ and let φ_{m+1} be the combination of the functions φ_G.

Now, since B is Baire, (1) and (3) yield a sequence $G_1 \supset G_2 \supset \cdots$, $G_m \in \mathcal{G}_m$ with

(5) $\bigcap_m G_m \neq \emptyset$.

By (4), $\varphi_1(G_1) \supset \varphi_2(G_2) \supset \cdots$, and since C is countably compact, there is $u \in C$ with

$$u \in \bigcap_m \varphi_m(G_m).$$

For each m, let us fix a finite set $A_m \subset B$ such that any $v \in C$ which coincides with u on A_m belongs to $\varphi_m(G_m)$. Since the Baire space B has no isolated points, nonempty open sets in B are uncountable. Therefore, by (6), B being also a P-space, there is

$$x \in \bigcap_m G_m \setminus \overline{\bigcup_m A_m}.$$

Let $v_0, v_1 \in C$ coincide with u on $\bigcup_m A_m$, $v_0(x) = 0$, $v_1(x) = 1$. Then, by (4),

$$(x, v_i) \in \bigcap_m G_m \times \varphi_m(G_m) \subset U \times W \setminus \bigcup_m F_m,$$

and $e(x, v_i) = i$, for $i = 0, 1$.

This ends the proof of the proposition. □

To complete a verification of the properties of B described in Example 1.4, let us notice that

$$B \text{ is not the union of } \aleph_1 \text{ nowhere dense subsets.} \tag{$*$}$$

We now show that if (CH) is assumed, B is a Namioka space. For this let $f : B \to C_p(K)$ be a continuous function with K compact. Let T be the subspace of B consisting of those elements with countable support. Notice that T is dense in B and the cardinality of T is $|T| = 2^{\aleph_0} = \aleph_1$. Express $f(T) = \{u_\alpha\colon \alpha < \omega_1\}$ and let H_ξ be the closed linear span of $\{u_\alpha\colon \alpha < \xi\}$. Since $H_1 \subseteq H_2 \subseteq \cdots$ and $C_p(K)$ has countable tightness, $H = \bigcup_{\xi < \omega_1} H_\xi$ is closed. In effect, $f(B) \subseteq H$ and by ($*$), there is a nonempty open set W in B with $W \subseteq f^{-1}(H_\xi)$, for some ξ. Then $f|_W : W \to H_\xi$ and H_ξ is norm-separable, being pointwise separable. In effect there is a point of continuity of $f|_W : W \to (H_\xi, \text{norm})$. This completes the proof of Example 1.4.

We remark that the space T (from the previous paragraph), consisting of those elements of B with countable support, was studied by Talagrand in [17]. He showed that T was a Choquet space which was not a Namioka space. Our proof that B is a Namioka space required the use of (CH). We do not know whether B is a Namioka space in every model of set theory. A natural question would be to ask what happens under $(\mathsf{MA} + \neg\mathsf{CH})$ (Martin's axiom plus the negation of (CH)).

We shall close this note with an observation related to some results obtained by Maxim Burke [3].

Let us recall that X is a Lindelöf Σ-space if X is a continuous image of a closed subset of the product of a metrizable separable space and a compact space.

Proposition 5.2. *Any separately continuous function $f : X \times C \to \mathbb{R}$, defined on the product of a Lindelöf Σ-space X and a countably compact space C, is Borel measurable. In particular, for any Baire subspace B of $X \times C$ the set of points of continuity of the restriction $f|B : B \to \mathbb{R}$ is dense in B.*

Proof. The proof is based on the following claim:

Claim. *If $f : X \times C \to \mathbb{R}$ is as in the theorem then there is a continuous surjection $v : C \to C^*$, where C^* is compact, and a continuous $u : X \to C_p(C^*)$ such that, with $e : C_p(C^*) \times C^* \to \mathbb{R}$ being the evaluation, $f(x,c) = e(u(x), v(c))$, and $u(X)$ separates the points in C^*.*

Proof. To that end, we define a continuous map $v : C \to C_p(X)$ by $v(c)(x) = f(x,c)$, and set $C^* = v(C)$. Then C^* is countably compact and a theorem of Baturov [1, Section III.6] shows, in fact, that C^* is actually compact. Now, define the map $u : X \to C_p(C^*)$ by $u(x)(z) = z(x)$. (That is, with $C^* \subseteq C_p(X)$ we interpret x in X as a function on C^*.) Then, $e(u(x), v(c)) = v(c)(x) = f(x,c)$. That shows the claim.

Now, using the Claim, we can justify the proposition as follows. The set $u(X)$ is a Lindelöf Σ-space and separates the points of C^*. Hence the space $C_p(C^*)$ is a Lindelöf Σ-space; cf. [1, IV.2.10]; therefore, it is descriptive; cf. Hansell [5]. Using the terminology of [3, Remark 5.13] we have that $C_p(C^*)$ is narrow, and hence, by [3, Proposition 5.19], the evaluation $e : C_p(C^*) \times C^* \to \mathbb{R}$ satisfies condition (b) in Proposition 2.3 in [3]. It is now clear from [3, Definition 2.1] that the map f, being the composition of the evaluation e and the continuous map $(u, v) : X \times C \to C_p(C^*) \times C^*$, is Borel measurable. □

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Further reading

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